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CONSTRUCTION OF NON-LINEAR SEMI-GROUPS USING PRODUCT FORMULAS

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ABSTRACT

Under certain circumstances, the Trotter–Lie formula $W_t = \lim(U_{t/n}V_{t/n})^n$ is used to construct a non-linear semi-group W_t on closed subsets of L^p, $1 \leqq p < \infty$. In particular we consider the situation where $U_t = e^{tA}$ is a positivity preserving C_0 (linear) semi-group and V_t is generated by a (non-linear) function F with certain monotonicity properties. In general, A and F are "singular" on L^p and no requirement is made that one of them be "relatively bounded" with respect to the other. The generator of the resulting semi-group W_t turns out to be an extension of $A + F$ restricted to a suitable domain.

1. Introduction

This paper is concerned with the convergence of the Trotter–Lie formula

$$W_t = \lim_{n\to\infty} (U_{t/n}V_{t/n})^n, \tag{1}$$

where U_t and V_t are non-linear semi-groups on a closed subset of a Banach space. Brezis and Pazy, [1, theor. 3.7 and 3.8] and [2, theor. 3.2 and corol. 4.3], have proved under various circumstances that (1), or a similar formula, in fact holds. They assume, however, that the semi-group W_t exists and is generated in the sense of Crandall and Liggett [4, theor. 1].

Our approach is to use (1) to help construct the semi-group W_t. This, of course, is done under some special circumstances. One can then go back and investigate the nature of the generator of W_t.

To be more specific, if K is a subset of a Banach space E, a semi-group on K is a collection of maps $U_t: K \to K$, $t \geqq 0$, satisfying:

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(2) $$U_t U_s = U_{t+s}, \qquad U_0 = \text{identity};$$

and

(3) for all $\phi \in K$, the curve $t \mapsto U_t\phi$ is continuous from $[0,\infty)$ into E.

If in addition there exists a real β such that

(4) $$\| U_t\phi - U_t\psi \| \leqq e^{\beta t} \| \phi - \psi \|$$

for all $t \geqq 0$ and $\phi, \psi \in K$, then U_t is called a *semi-group of type* β.

We are interested in the case where $E = L^p(\mu)$, $1 \leqq p < \infty$, for a positive measure μ on a set X. In this paper L^p denotes equivalence classes of real functions with finite p-norm, (except in Section 6, where the complex case is discussed). For real measurable functions ϕ and ψ, $\phi \leqq \psi$ means $\phi(x) \leqq \psi(x)$ a.e. $[\mu]$. In the next section we prove the following result.

THEOREM 1. *Let μ be a positive measure on X. For some $p \in [1,\infty)$, let K be a closed subset of $L^p(\mu)$ such that $|\phi| \in K$ whenever $\phi \in K$. Suppose U_t and V_t are semi-groups on K of types β and γ respectively such that*

(a) *if $\phi, \psi \in K$ with $|\phi| \leqq \psi$, then $|U_t\phi| \leqq U_t\psi$;*

(b) *there is a $\lambda \geqq 0$ such that $|V_t\phi| \leqq e^{\lambda t}|\phi|$ for all $\phi \in K$;*

(c) *$U_t(c\phi) \leqq cU_t\phi$ for all $\phi \geqq 0$ in K and $c \geqq 1$ (this condition is unnecessary if $\lambda = 0$).*

Assume also that there is a dense subset K_0 of K such that

(d) *for all $\phi \in K_0$ and $t \geqq 0$, $\lim_{n\to\infty}[(U_{t/n}V_{t/n})^n\phi](x) = (W_t\phi)(x)$ exists a.e. $[\mu]$;*

(e) *for all $\phi \in K_0$ and sequences $t_m \downarrow 0$, $\lim_{m\to\infty}(W_{t_m}\phi)(x) = \phi(x)$ a.e. $[\mu]$.*

Then W_t extends to a semi-group on K of type $\beta + \gamma$ and

(5) $$W_t\phi = L^p - \lim_{n\to\infty} (U_{t/n}V_{t/n})^n\phi$$

for all $\phi \in K$ and $t \geqq 0$.

The proof involves some careful applications of the Lebesgue dominated convergence theorem, made possible by conditions (a), (b) and (c) above, and also uses a result of Chernoff [3] on product limits. Observe that in the case where $U_t = e^{tA}$ is a linear semi-group, condition (c) is automatic and condition (a) says that e^{tA} is positivity preserving.

In the subsequent sections we discuss a wide class of examples to which Theorem 1 easily applies. In particular, we have the following corollary to Theorem 2 in Section 4.

COROLLARY. *Let μ be a positive Radon measure on a locally compact Hausdorff space X. Let $f(t)$ be a polynomial such that $f(0)=0$ and $f(t)<0$ for all sufficiently large $t>0$; and let $(F\phi)(x)=f(\phi(x))$. Suppose e^{tA} is a positivity preserving C_0 (linear) semi-group on both $L^p(\mu)$ and $C_0(X)$ with $\|e^{tA}\|_p \leqq e^{\beta t}$ and $\|e^{tA}\|_\infty \leqq e^{\beta t}$. Then there exists a semi-group W_t on K_p, the positive cone in L^p, whose generator is an extension of $A+F$ defined on $D_p(A)\cap K_{mp}$, where m is the degree of the polynomial $f(t)$.*

Marsden [6] has also used the Trotter–Lie formula to construct semi-groups. The situation he is concerned with, however, is rather different than ours; and he shows convergence of (1) only on a finite t-interval, (see [6, theor. 5.1, p. 67]).

The Trotter–Lie formula has also been helpful in proving a linear perturbation theorem for hypercontractive semi-groups in quantum field theory. See [9]. The results in the present paper are partially inspired by those techniques.

I wish to thank Paul Chernoff for suggesting this line of inquiry to me.

2. Proof of Theorem 1

We begin with a lemma.

LEMMA 1. *Under the hypotheses of Theorem 1, it follows that for all $\phi\in K$,*

$$|(U_tV_t)^k\phi|\leqq e^{\lambda tk}U_{tk}(|\phi|) \tag{6}$$

for all $t\geqq 0$ and $k=0,1,2,\cdots$. In particular,

$$|(U_{t/n}V_{t/n})^n\phi|\leqq e^{\lambda t}U_t(|\phi|) \tag{7}$$

for all $t\geqq 0$ and $n=1,2,3,\cdots$.

PROOF. We prove (6) by induction on k. For $k=0$ it is trivial. Assume, therefore, that (6) is true for a specific k. Let $\psi=(U_tV_t)^k\phi$. Then $|V_t\psi|\leqq e^{\lambda t}|\psi|$; and by the induction hypothesis $|\psi|\leqq e^{\lambda tk}U_{tk}(|\phi|)$. Thus $|V_t\psi|\leqq e^{\lambda t(k+1)}U_{tk}(|\phi|)$. From conditions (a) and (c) in Theorem 1, it now follows that $|U_tV_t\psi|\leqq U_t[e^{\lambda t(k+1)}U_{tk}(|\phi|)]\leqq e^{\lambda t(k+1)}U_{t(k+1)}(|\phi|)$. This proves the lemma.

In proving Theorem 1, we will use a slightly strengthened version of the dominated convergence theorem. The result we use follows from theorem 16, chapter 4, of Royden [7] by a subsequence argument.

MODIFIED DOMINATED CONVERGENCE THEOREM. *Let μ be a positive measure on X; and let $\{f_m\}$, $m=1,2,3,\cdots$, and f be measurable functions on X such that $f_m(x)\to f(x)$ a.e. $[\mu]$. Suppose there exist non-negative measurable functions*

$\{g_m\}$, $m = 1,2,3,\cdots$, and g on X with $|f_m(x)| \leqq g_m(x)$ a.e. $[\mu]$ and $g_m \to g$ in $L^p(\mu)$, for some $p \in [1,\infty)$. Then $f_m \to f$ in $L^p(\mu)$.

We shall also use the following result of Chernoff on product limits and semi-groups.

THEOREM. (Chernoff [3], theorem 2.5.3) *Let M be a complete metric space. Let F_t, $t > 0$, be a collection of Lipschitz mappings on M such that for each $\phi \in M$ the curve $t \mapsto F_t\phi$ is continuous. Assume:*

(i) *For each $t > 0$ and $\phi \in M$, $\lim_{n\to\infty}(F_{t/n})^n\phi = G_t\phi$ exists.*

(ii) *$\|F_t\|_{\mathrm{Lip}}$ is locally bounded.*

(iii) *For each $t > 0$, $\sup_n \|(F_{t/n})^n\|_{\mathrm{Lip}} < \infty$.*

Then $G_{t+s} = G_tG_s$ for all $t, s > 0$; and for all $\phi \in M$ the curve $t \mapsto G_t\phi$ is continuous.

PROOF OF THEOREM 1. We first show that the product limit formula (5) holds for all $\phi \in K_0$. For such ϕ, $W_t\phi$ is the almost everywhere limit of $(U_{t/n}V_{t/n})^n\phi$ as $n \to \infty$. By the lemma the $(U_{t/n}V_{t/n})^n\phi$ are all dominated by $e^{\lambda t}U_t(|\phi|)$, which is in L^p. Thus the convergence is in the L^p norm.

Consequently, for ϕ and ψ in K_0,

$$\|W_t\phi - W_t\psi\|_p = \lim_{n\to\infty}\|(U_{t/n}V_{t/n})^n\phi - (U_{t/n}V_{t/n})^n\psi\|_p$$

$$\leqq e^{(\beta+\gamma)t}\|\phi - \psi\|_p.$$

Thus W_t extends to a map on all of K satisfying the same inequality.

The product limit formula (5) is now easily seen to hold for all $\phi \in K$. Indeed, the maps $(U_{t/n}V_{t/n})^n$, $n = 1,2,3,\cdots$, are uniformly Lipschitz on K and converge (pointwise in L^p norm) on the dense subset K_0 to the Lipschitz map W_t. Hence, the convergence is on all of K.

We now apply Chernoff's theorem with $F_t = U_tV_t$. We have just shown that condition (i) is met. Conditions (ii) and (iii) are immediate since U_t and V_t are of types β and γ. We conclude that W_t satisfies the semi-group property (2) and that for all $\phi \in K$, $t \mapsto W_t\phi$ is continuous for $t > 0$.

It remains to show continuity at $t = 0$. Let $t_m \downarrow 0$. Suppose first that $\phi \in K_0$. Then by assumption ϕ is the almost everywhere limit of $W_{t_m}\phi$ as $m \to \infty$. Taking the L^p limit in (7), we see that $|W_{t_m}\phi| \leqq \exp(\lambda t_m)U_{t_m}(|\phi|)$. Since $\exp(\lambda t_m)U_{t_m}(|\phi|) \to |\phi|$ in L^p, it follows from the modified version of the dominated convergence theorem that $W_{t_m}\phi \to \phi$ in L^p. The same is now true for an arbitrary ϕ in K since the W_{t_m} are uniformly Lipschitz as $m \to \infty$.

This concludes the proof of Theorem 1.

3. A non-linear product formula

Our ultimate goal is to use Theorem 1 to show that certain semi-groups on $C_0(X)$, the continuous functions vanishing at ∞, extend to semi-groups on $L^p(\mu)$. In this section we prove a product formula which will enable us to verify conditions (d) and (e) in Theorem 1. The results of this section are formulated on an arbitrary Banach space E.

Let $U_t = e^{tA}$ be a C_0 (linear) semi-group with $\|e^{tA}\| \leqq e^{\beta t}$. Let $F: E \to E$ be semi-Lipschitz (i.e. Lipschitz when restricted to any bounded set in E). It is well known (Segal [8], theor. 1) that there is a maximal continous semi-flow W_t on E satisfying the integral equation

$$W_t\phi = e^{tA}\phi + \int_0^t e^{(t-\tau)A}F(W_\tau\phi)d\tau. \tag{8}$$

We say *semi-flow* (instead of semi-group) because the (continuous) curve $t \mapsto W_t\phi$ might only be defined on an interval $[0, T_\phi)$. However, if the endpoint T_ϕ is finite, we must have $\lim_{t\uparrow T_\phi}\|W_t\phi\| = \infty$. In other words, the only way $W_t\phi$ can fail to exist for all $t \geqq 0$ is for it to blow up at a finite $t = T_\phi$.

Also, there is the maximal continuous semi-flow V_t on E satisfying

$$V_t\phi = \phi + \int_0^t F(V_\tau\phi)d\tau. \tag{9}$$

Observe that the generator of W_t is precisely $A + F$, i.e. $\lim_{t\downarrow 0} t^{-1}(W_t\phi - \phi)$ exists if and only if $\phi \in D(A)$ and in that case the limit equals $A\phi + F(\phi)$. The generator of V_t is F.

LEMMA 2. (preliminary calculation) *Let W_t and V_t be as above. Then*

$$e^{tA}V_t\psi - W_t\psi = \int_0^t [e^{tA}F(V_\tau\psi) - e^{(t-\tau)A}F(W_\tau\psi)]d\tau.$$

PROOF. Write $e^{tA}V_t\psi - W_t\psi$ as $e^{tA}[V_t\psi - \psi] - [W_t\psi - e^{tA}\psi]$ and use the integral equations (8) and (9).

PROPOSITION 1. *Under the above circumstances, suppose in addition that V_t is a semi-group on E of type γ. Assume also that $F(0) = 0$. Then W_t is a semi-group on E of type $\beta + \gamma$. Moreover, for all $\phi \in E$ and $T > 0$*

$$\lim_{n\to\infty} (e^{t/nA}V_{t/n})^n\phi = W_t\phi \tag{10}$$

uniformly for $t \in [0, T]$.

REMARKS. This proposition follows from corollary 4.3 in Brezis and Pazy [2], at least if E is uniformly convex. (See also corollary 10.3 in Kato [5].) Our proof does not depend on the advanced non-linear semi-group theory, but rather is based on the integral equations in the form of Lemma 2. Moreover, the calculation below can be used to prove (10) in cases where the semi-group theory does not apply. For example, if V_t is only a continuous semi-flow (and not a semi-group), then for each $\phi \in E$ the product formula (10) still holds uniformly on *some* interval $[0, T]$. If e^{tA} is a contraction semi-group, then we may drop the requirement $F(0) = 0$ and still have (10) hold uniformly on some interval.

PROOF. For a given $\phi \in E$ let $[0, T_\phi)$ be the interval on which $W_t\phi$ is defined. We first show that for $T < T_\phi$ the product formula (10) holds uniformly on $[0, T]$. Expanding in a telescoping sum, we see that for any $t \in [0, T]$,

$$\begin{aligned}
\|(e^{t/nA}V_{t/n})^n\phi - W_t\phi\| &= \Big\| \sum_{k=0}^{n-1} [(e^{t/nA}V_{t/n})^{n-k-1} e^{t/nA}V_{t/n}(W_{t/n})^k\phi \\
&\qquad - (e^{t/nA}V_{t/n})^{n-k-1} W_{t/n}(W_{t/n})^k\phi] \Big\| \\
&\leqq \sum_{k=0}^{n-1} e^{(\beta+\gamma)t(n-k-1)/n} \| e^{t/nA}V_{t/n}W_{tk/n}\phi - W_{t/n}W_{tk/n}\phi \| \\
&\leqq e^{(\beta+\gamma)T} \sum_{k=0}^{n-1} \| e^{t/nA}V_{t/n}W_{tk/n}\phi - W_{t/n}W_{tk/n}\phi \| \\
&\leqq ne^{(\beta+\gamma)T} \sup\{\| e^{t/nA}V_{t/n}W_s\phi - W_{t/n}W_s\phi \| : \\
&\qquad\qquad 0 \leqq s \leqq (n-1)T/n\}.
\end{aligned}$$

Next we use the preliminary calculation of Lemma 2. For notational simplicity we let

$$g(t, \tau, s) = e^{tA}F(V_\tau W_s\phi) - e^{(t-\tau)A}F(W_\tau W_s\phi).$$

Then the previous expression is bounded by

$$\begin{aligned}
&ne^{(\beta+\gamma)T} \sup\left\{ \int_0^{t/n} \|g(t/n, \tau, s)\| d\tau : 0 \leqq s \leqq (n-1)T/n \right\} \\
&\quad \leqq te^{(\beta+\gamma)T} \sup\{\|g(t/n, \tau, s)\| : 0 \leqq s \leqq (n-1)T/n, 0 \leqq \tau \leqq t/n\} \\
&\quad \leqq Te^{(\beta+\gamma)T} \sup\{\|g(t, \tau, s)\| : 0 \leqq s \leqq (n-1)T/n, 0 \leqq \tau \leqq t \leqq T/n\}.
\end{aligned}$$

This last expression converges to 0 as $n \to \infty$ since $g(t, \tau, s)$ is jointly continuous in all three variables and $g(0, 0, s) = 0$ for all s. This proves (10) for $T < T_\phi$.

From (10) it now follows that $\| W_t\phi - W_t\psi \| \leqq e^{(\beta+\gamma)t} \| \phi - \psi \|$ whenever $t < \min(T_\phi, T_\psi)$. Furthermore, since $F(0) = 0$, we must have $W_t(0) = 0$ for all $t \geqq 0$. Thus $\| W_t\phi \| \leqq e^{(\beta+\gamma)t} \| \phi \|$ whenever $t < T_\phi$. Consequently $W_t\phi$ cannot blow up at a finite value of t, and so $T_\phi = \infty$. Thus W_t is a semi-group, and the proposition is proved.

4. A class of examples

In this section X denotes a locally compact Hausdorff space. $C_c(X)$ denotes the real continuous functions on X with compact support and $C_0(X)$ the real continuous functions on X vanishing at infinity. We consider $C_0(X)$ as a Banach space with the sup norm $\| \ \|_\infty$. Also, μ denotes a positive Radon measure on X, i.e. a Borel measure determined by a positive linear functional on $C_c(X)$ in the usual way. The crucial fact we shall use is that $C_c(X)$ is dense in $L^p = L^p(\mu)$ for all $p \in [1, \infty)$.

Our goal is to construct a semi-group on L^p which in some sense is generated by $A + F$, where A is the generator of a C_0 (linear) semi-group and F is some "singular" non-linear mapping in L^p, for example a polynomial. The following elementary proposition describes the class of functions F (and the semi-groups generated by F) which we consider.

PROPOSITION 2. *For real functions ϕ on X let $F(\phi)$ be given by $F(\phi)(x) = f(x, \phi(x))$, where $f: X \times R \to R$ is continuous and satisfies*:

(a) $f(x, 0) = 0$ *for all* $x \in X$.

(b) *For any $\alpha > 0$ the functions $f(x, \cdot)$ are uniformly Lipschitz on $[-\alpha, \alpha]$, i.e. there exists $c(\alpha)$ such that $|f(x,t) - f(x,s)| \leqq c(\alpha)|t - s|$ whenever t, $s \in [-\alpha, \alpha]$.*

(c) *There exists $\gamma \in R$, independent of x, such that $t \mapsto f(x,t) - \gamma t$ is monotone non-increasing on R for all $x \in X$.*

Then $F: C_0(X) \to C_0(X)$ is semi-Lipschitz; and the maximal continuous semi-flow V_t on $C_0(X)$ generated by F, i.e. satisfying the integral equation (9), *is in fact a semi-group of type γ on $C_0(X)$. V_t also satisfies*:

(i) $| V_t\phi | \leqq e^{\gamma t} | \phi |$;

(ii) $V_t\phi \geqq 0$ *whenever* $\phi \geqq 0$.

Furthermore, for all $p \in [1, \infty)$ V_t extends to a semi-group of type γ on L^p (also denoted by V_t) satisfying (i) *and* (ii).

PROOF. Denote $f(x, \cdot)$ by f_x. Then for $\phi, \psi \in C_0(X)$

$$|F\phi(x) - F\psi(x)| = |f_x(\phi(x)) - f_x(\psi(x))| \leqq c(\alpha)|\phi(x) - \psi(x)| \tag{11}$$

where $\alpha = \max[\|\phi\|_\infty, \|\psi\|_\infty]$. This proves $F: C_0(X) \to C_0(X)$ is semi-Lipschitz. Let V_t be the corresponding semi-flow on $C_0(X)$ satisfying the integral equation (9). Then for all $x \in X$

$$(V_t\phi)(x) = \phi(x) + \int_0^t f_x(V_\tau\phi(x))d\tau.$$

Since $f_x(0) = 0$, it follows that if $(V_t\phi)(x) = 0$ for some t, then $(V_s\phi)(x) = 0$ for all $s \geqq t$. Thus $(V_t\phi)(x)$ never has the opposite sign from $\phi(x)$. This proves (ii).

Furthermore, since $f_x(t) - \gamma t$ is non-increasing, it follows that

$$|V_t\phi(x) - V_t\psi(x)| \leqq e^{\gamma t}|\phi(x) - \psi(x)| \tag{12}$$

for all $\phi, \psi \in C_0(X)$, $x \in X$, and $t \geqq 0$. In particular, since $V_t(0) = 0$,

$$|V_t\phi(x)| \leqq e^{\gamma t}|\phi(x)|. \tag{13}$$

This proves (i) and shows that V_t is a semi-group, which (because of (12)) is of type γ.

The extension to L^p is straightforward using (12) and (13), and the dominated convergence theorem for L^p continuity of $t \mapsto V_t\phi$.

THEOREM 2. *Let F and V_t be as in the previous proposition. Fix $p \in [1, \infty)$ and let e^{tA} be a positivity preserving C_0 semi-group on both L^p and $C_0(X)$ with $\|e^{tA}\|_p \leqq e^{\beta t}$ and $\|e^{tA}\|_\infty \leqq e^{\beta t}$. Let W_t be the semi-group of type $\beta + \gamma$ on $C_0(X)$ satisfying the integral equation* (8) *as described in Proposition* 1.

(i) *Then W_t extends to a semi-group of type $\beta + \gamma$ on L^p with*

$$W_t\phi = L^p - \lim_{n\to\infty} (e^{t/nA}V_{t/n})^n\phi \tag{14}$$

for all $\phi \in L^p$ and $t \geqq 0$. Moreover, W_t preserves the positive cone in L^p.

(ii) *Let B be the generator of W_t in L^p with domain $D_p(B)$, i.e. $B\phi = L^p - \lim_{t\downarrow 0} t^{-1}(W_t\phi - \phi)$ for all those ϕ in L^p for which the limit exists. Then $D_p(B) \cap C_0(X) = D_p(A) \cap C_0(X)$ and $B\phi = A\phi + F\phi$ for all $\phi \in D_p(A) \cap C_0(X)$. ($D_p(A)$ denotes the domain of A as the generator of e^{tA} in L^p.)*

Furthermore, if $p > 1$ W_t leaves $D_p(A) \cap C_0(X)$ invariant. In particular, if $\phi \in D_p(A) \cap C_0(X)$, the curve $t \mapsto W_t\phi$ is right differentiable in L^p for all $t \geqq 0$ with right derivative $AW_t\phi + F(W_t\phi)$.

(iii) *Suppose for some $q \in (p, \infty)$ $F: L^p \cap L^q \to L^p$ is semi-Lipschitz (as is the case if F is a polynomial). Then the integral equation* (8) *holds for all $\phi \in L^p \cap L^q$. (The integrand is continuous in L^p.) Moreover, $D_p(B) \cap L^q = D_p(A) \cap L^q$ and $B\phi = A\phi + F\phi$ for all $\phi \in D_p(A) \cap L^q$.*

Furthermore, if $p > 1$ W_t leaves $D_p(A) \cap L^q$ invariant. In particular, if $\phi \in D_p(A) \cap L^q$ the curve $t \mapsto W_t\phi$ is right differentiable in L^p for all $t \geqq 0$ with right derivative $AW_t\phi + F(W_t\phi)$.

PROOF. We apply Theorem 1 with $K = L^p$ and $K_0 = L^p \cap C_0(X)$. $U_t = e^{tA}$ is linear and positivity preserving and thus satisfies conditions (a) and (c). Proposition 2 says that V_t satisfies condition (b). It follows from Proposition 1 that for all ϕ in K_0

$$W_t\phi = C_0(X) - \lim_{n\to\infty} (e^{t/nA} V_{t/n})^n \phi$$

uniformly for compact intervals of t. Thus condition (d) is easily satisfied. Similarly, condition (e) is satisfied since W_t is a semi-group on $C_0(X)$.

Consequently, Theorem 1 implies that W_t extends to a semi-group of type $\beta + \gamma$ on L^p satisfying (14). Since both e^{tA} and V_t leave the positive cone in L^p invariant (recall (ii) of Proposition 2), (14) implies that W_t must do likewise. This proves the first part of the theorem.

Turning to the second part, note first that for $\phi, \psi \in L^p \cap C_0(X)$ formula (11) implies

$$\|F\phi - F\psi\|_p \leqq c(\alpha)\|\phi - \psi\|_p \tag{15}$$

where $\alpha = \max[\|\phi\|_\infty, \|\psi\|_\infty]$. Thus, for $\phi \in L^p \cap C_0(X)$ the integrand in

$$W_t\phi = e^{tA}\phi + \int_0^t e^{(t-\tau)A}F(W_\tau\phi)d\tau \tag{16}$$

is continuous in L^p and

$$L^p - \lim_{t\downarrow 0} t^{-1}\int_0^t e^{(t-\tau)A}F(W_\tau\phi)d\tau = F\phi. \tag{17}$$

It follows immediately that $D_p(B) \cap C_0(X) = D_p(A) \cap C_0(X)$ and $B\phi = A\phi + F\phi$ for $\phi \in D_p(A) \cap C_0(X)$.

Now if $\phi \in D_p(A) \cap C_0(X)$, then certainly $W_s\phi \in L^p \cap C_0(X)$. If $p > 1$, we must show $W_s\phi \in D_p(A)$. For $p > 1$ L^p is reflexive; and it therefore suffices to show that $t^{-1}\|e^{tA}W_s\phi - W_s\phi\|_p$ remains bounded as $t \downarrow 0$. But by (16) and (17) that is the same as showing $t^{-1}\|W_tW_s\phi - W_s\phi\|_p$ remains bounded as $t \downarrow 0$. Since $t^{-1}\|W_tW_s\phi - W_s\phi\|_p = t^{-1}\|W_sW_t\phi - W_s\phi\|_p \leqq e^{(\beta+\gamma)s}t^{-1}\|W_t\phi - \phi\|_p$, the result follows because $\phi \in D_p(B)$.

For the third part of the theorem, first observe that by interpolation e^{tA} is a positivity preserving C_0 semi-group on L^q with $\|e^{tA}\|_q \leqq e^{\beta t}$. Thus, W_t is a semi-group of type $\beta + \gamma$ on L^q just as it is on L^p. To show that the integral equation holds for $\phi \in L^p \cap L^q$, choose a sequence $\phi_m \in C_0(X)$ with $\phi_m \to \phi$ in both L^p and L^q and apply the dominated convergence theorem to the L^p valued integrals $\int_0^t e^{(t-\tau)A} F(W_\tau \phi_m) d\tau$. The rest of the proof is virtually the same as for the previous part of the theorem, and we omit the details.

This concludes the proof of Theorem 2.

5. Remarks

In the proof of Theorem 1, Chernoff's theorem was used to show that W_t satisfied the semi-group property (2) and was strongly continuous for $t > 0$. If in proving Theorem 2 we were to repeat the arguments in Theorem 1 rather than simply apply it, we would not need Chernoff's result. That $W_t W_s = W_{t+s}$ on L^p follows since it is true on $C_0(X)$ and continuity of $t \mapsto W_t \phi$ can be proved for $t > 0$ the way it was proved for $t = 0$.

Note further that although the product formula $W_t \phi = \lim (e^{t/nA} V_{t/n})^n \phi$ in $C_0(X)$ holds uniformly on compact t-intervals, we have not shown the same to be true in L^p. Convergence in L^p for each t is all that has been asserted.

An interesting feature of Theorem 2 is that we have not required F to be "relatively bounded" with respect to A in some sense. The "degrees of singularity" of A and F as mappings in L^p are independent of each other.

Finally, observe that the behavior of W_t on the positive cone in L^p depends solely on the values of $f(x, t)$ for $t \geqq 0$. Thus the corollary stated in Section 1 follows easily from Theorem 2.

6. Extension to complex L^p spaces

The results above are stated for real L^p spaces; and the main example, i.e. when $F(\phi)$ is a polynomial, fits naturally into this context. However, one can go back and see how the results carry over to the complex L^p spaces.

First of all, if we understand that for arbitrary complex functions the expression $\phi \leqq \psi$ means that ϕ and ψ are real valued with $\phi(x) \leqq \psi(x)$ a.e. $[\mu]$, then Theorem 1 remains valid as stated (with the same proof) for complex L^p spaces. On the other hand, to construct V_t as in Proposition 2 on the complex spaces $C_0(X)$ and $L^p(\mu)$ requires a modification. For complex functions ϕ on X let $F(\phi)(x) = f(x, \phi(x))$, where $f: X \times C \to C$ is continuous and satisfies:

(a) $f(x, 0) = 0$ for all $x \in X$.

(b) For any $\alpha > 0$ there exists $c(\alpha)$ such that $|f(x, z) - f(x, w)| \leqq c(\alpha) |z - w|$ whenever $|z|, |w| \leqq \alpha$.

(c) There exists $\gamma \in R$ such that for all $x \in X$, $z \mapsto f(x, z) - \gamma z$ is a dissipative operator on the two dimensional real Hilbert space C.

With this F the results of Proposition 2 and Theorem 2 are valid for the complex spaces $C_0(X)$ and $L^p(\mu)$, except that V_t and W_t need not preserve the positive cones.

An example of such an f can be constructed as follows: Let $g: X \times R \to R$ satisfy the hypotheses of Proposition 2. Then it is straightforward to verify that $f: X \times C \to C$ given by $f(x, z) = (\operatorname{sgn} z) g(x, |z|)$ satisfies the requirements stated above.

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